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Chromatic Number and the 2-Rank of a Graph

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We show that if the adjacency matrix of a graph X has 2-rank $2r$, then the

1. INTRODUCTION

The rank of the adjacency matrix of a graph over the real field is a natural parameter associated with the graph, and there has been substantial interest in relating this to other graph theoretic parameters. In particular the relationship between the rank and chromatic number of a graph, and the rank and the maximum number of vertices of a graph has been extensively studied. This was at least in part motivated by a conjecture of Van Nuffelen [19] that

$$\chi(X) \leqslant \mathrm{rk}(X),$$

where $\chi(X)$ is the chromatic number of the graph. This conjecture was proved incorrect by Alon and Seymour [1] who noted that if X is the complement of the folded 7-cube, then X has rank 29 and chromatic number 32. As it is clear that

$$\chi(X) \leqslant |V(X)| \leqslant 2^{\mathrm{rk}(X)}$$

for a reduced graph X, this leaves open question of determining the smallest function of the rank bounding the chromatic number of a graph. Various authors have pursued this question—see Razborov [11] and Raz and Spieker [10]. Kotlov and Lovasz [6] bound the number of vertices of a reduced graph in terms of the rank, and Kotlov [7] uses this bound to get an improved bound on the chromatic number in terms of the rank.

In this paper we consider the analogous problem for the binary rank of the adjacency matrix of a graph.

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2. REDUCED GRAPHS OF RANK $2r$

Let X be a finite simple graph (that is, no loops or multiple edges) with adjacency matrix $A(X)$. The 2-rank of X is the rank of $A(X)$ calculated over $GF(2)$, and denoted $rk_2(X)$. The following results from linear algebra are well known.

PROPOSITION 2.1. *If A is a symmetric matrix with zero diagonal, then $rk_2(A)$ is even.*

PROPOSITION 2.2. *If A is a symmetric matrix with rank k, then it has a $k \times k$ principal submatrix of rank k.*

If a graph X has two vertices with identical neighbourhoods, then we can delete one of the two vertices without changing either the rank or the chromatic number of X. Conversely, we can duplicate a vertex arbitrarily often without changing the rank or the chromatic number. In addition, we can add isolated vertices without changing the rank or chromatic number. This motivates the following definition.

DEFINITION 2.3. A graph X is reduced if it has no isolated vertices and the neighbourhoods of distinct vertices are distinct.

Note that some other authors use "reduced" purely for the condition that distinct vertices have distinct neighbourhoods.

Henceforth we assume that X is a reduced graph with 2-rank $2r$. Therefore the adjacency matrix of X has the form

$$A(X) = \begin{pmatrix} M & H^T \\ H & N \end{pmatrix},$$

where M is the adjacency matrix of a reduced graph on $2r$ vertices with 2-rank $2r$. Every row of H is a linear combination of the rows of M, and so there is some matrix R such that

$$H = RM.$$

The rows of N are formed from the same linear combinations of the rows of H^T and so

$$N = RH^T = RMR^T,$$

from which it follows that

$$A(X) = \begin{pmatrix} M & MR^T \\ RM & RMT^T \end{pmatrix} = \begin{pmatrix} I \\ R \end{pmatrix} M(I \quad R^T). \tag{1}$$

We note that this argument actually provides a simple proof of Proposition 2.2, in that it holds unchanged if we start by taking M to be the principal submatrix determined by any set of $2r$ rows that generates the row-space of $A(X)$.

The expression for the adjacency matrix of X given by Eq. 1 yields a useful representation of the graph X. If we let x_i denote the ith column of the matrix (IR^T), then X can be taken to have vertex set

$$V(X) = \{x_1, ..., x_n\},$$

with adjacency given by

$$x_i \sim x_j \qquad \text{if and only if} \qquad x_i^T M x_j = 1.$$

As M is an invertible symmetric matrix with zero diagonal, it determines a non-degenerate symplectic form $B(x, y)$ on $GF(2)^{2r}$ such that for all vectors $x, y \in GF(2)^{2r}$ the value of the form is given by

$$B(x, y) = x^T M y.$$

In a vector space with a form B, two vectors are said to be *orthogonal* if $B(x, y) = 0$. Therefore X is the non-orthogonality graph of a subset of $GF(2)^{2r}$ equipped with a non-degenerate symplectic form.

3. THE UNIVERSAL GRAPH

As X is reduced, every column of $(I \ \ R^T)$ is non-zero and no two columns are equal. Therefore X has at most $2^{2r} - 1$ vertices, and this occurs precisely when $(I \ \ R^T)$ has every non-zero vector of $GF(2)^{2r}$ as its columns. So, what graphs do we obtain with this maximum number of vertices? At first sight, there may be many possible graphs, depending on the choice of M. However we have the following standard result from finite geometry (see for example Taylor [17]).

THEOREM 3.4. *Let V be a vector space of even dimension $2r$ over $GF(2)$. Up to isometry, there is a unique non-degenerate symplectic form B which can be taken to be*

$$B(x, y) = x_0 y_1 + x_1 y_0 + \cdots + x_{2r-2} y_{2r-1} + x_{2r-1} y_{2r-2}.$$

DEFINITION 3.5. For every $r > 0$, define the universal graph $Y(2r)$ to have vertex set $GF(2)^{2r} \backslash 0$ where x is adjacent to y if and only if $B(x, y) = 1$.

THEOREM 3.6. *Every reduced graph X with 2-rank $2r$ is an induced subgraph of the universal graph $Y(2r)$.*

Proof. The discussion at the end of the previous section established that X is the non-orthogonality graph of a subset $\Omega \subseteq GF(2)^{2r} \backslash 0$, with respect to a non-degenerate symplectic form B. By Theorem 3 we can take B to be the particular form given above; then X is the subgraph of $Y(2r)$ induced by Ω. ∎

The graph $Y(2r)$ is well-known. It is the complement of the *symplectic* graph $Sp(2r, 2)$ which is the orthogonality graph of the unique non-degenerate symplectic form.

4. CHROMATIC NUMBER

To complete our main result, we merely need to establish the chromatic number of $Y(2r)$. We start by finding a bound on the size of an independent set in $Y(2r)$.

PROPOSITION 4.7 (Hoffman [4]). *If X is a regular graph on n vertices with valency k and minimum eigenvalue $\theta_{\min}$, then the maximum size $\alpha(X)$ of an independent set satisfies*

$$\alpha(X) \leqslant \frac{n}{1 - \dfrac{k}{\theta_{\min}}}.$$

The eigenvalues of $Y(2r)$ are known (see Seidel [14]), as it is a strongly regular graph with parameters

$$(2^{2r} - 1, 2^{2r-1}, 2^{2r-2}, 2^{2r-2})$$

and thus has eigenvalues

$$2^{2r-1}, 2^{r-1} \text{ and } -2^{r-1}.$$

Applying this to $Y(2r)$ we get that

$$\alpha(Y(2r)) \leqslant 2^r - 1,$$

and hence

$$\chi(Y(2r)) \geqslant 2^r + 1.$$

A subspace $W \subseteq GF(2)^{2r}$ is called totally *isotropic* if $B(u, v) = 0$ for all $u, v \in W$. If B is the form given above, then the maximal totally isotropic subspaces all have dimension r. As such a subspace contains $2^r - 1$ non-zero vectors it forms a maximum independent set in $Y(2r)$. A collection of maximal totally isotropic subspaces of $GF(2)^{2r}$ that intersect only in $\mathbf{0}$ is called a *symplectic spread.* Symplectic spreads exist for all $r \geqslant 2$ (see Thas [18]), and so there is a colouring of $Y(2r)$ with $2^r + 1$ colour classes, showing that the chromatic number of $Y(2r)$ is exactly $2^r + 1$.

Therefore we obtain our main theorem.

THEOREM 4.8. *Let X be a reduced graph with 2-rank $2r$. Then X is an induced subgraph of the graph $Y(2r)$ and hence $\chi(X) \leqslant 2^r + 1$.*

An immediate consequence of this is that any graph (reduced or otherwise) of 2-rank $2r$ has chromatic number at most $2^r + 1$.

5. EQUALITY

The bound on the chromatic number given in the theorem can obviously be much too large; in general it will exceed the number of vertices of the graph. Nevertheless the bound is tight for a large number of graphs, and in this section we identify two interesting graphs which meet this bound.

There are two non-degenerate quadratic forms on $GF(2)^{2r}$. The *hyperbolic* form may be taken to be

$$Q^+(x) = x_0 x_1 + \cdots + x_{2r-2} x_{2r-1},$$

and the *elliptic* form to be

$$Q^-(x) = x_0^2 + x_0 x_1 + x_1^2 + x_2 x_3 + \cdots + x_{2r-2} x_{2r-1}.$$

Each of these forms partitions the vectors of $GF(2)^{2r}$ into the *singular* vectors u such that $Q(u) = 0$ and the *non-singular* vectors with $Q(u) = 1$. We will consider the following four induced subgraphs of $Y(2r)$: $S^+(2r)$ and

TABLE I

Graph	n	k	λ	μ
$Y(2r)$	$2^{2r}-1$	2^{2r-1}	2^{2r-2}	2^{2r-2}
$S^+(2r)$	$2^{2r-1}+2^{r-1}-1$	2^{2r-2}	$2^{2r-3}-2^{r-2}$	2^{2r-3}
$S^-(2r)$	$2^{2r-1}-2^{r-1}-1$	2^{2r-2}	$2^{2r-3}+2^{r-2}$	2^{2r-3}
$N^+(2r)$	$2^{2r-1}-2^{r-1}$	$2^{2r-2}-2^{r-1}$	$2^{2r-3}-2^{r-2}$	$2^{2r-3}-2^{r-1}$
$N^-(2r)$	$2^{2r-1}+2^{r-1}$	$2^{2r-2}+2^{r-1}$	$2^{2r-3}+2^{r-2}$	$2^{2r-3}+2^{r-1}$

$S^-(2r)$ are the graphs induced by the singular points of the hyperbolic and elliptic forms respectively, and $N^+(2r)$ and $N^-(2r)$ the graphs induced by the non-singular points. (We warn the reader that similar terminology is sometimes used in the literature to refer to the *complements* of these graphs.)

All four of these graphs are strongly regular, with parameters as shown in Table I.

THEOREM 5.9. *The graph $S^-(2r)$ has chromatic number 2^r+1.*

Proof. The eigenvalues of $S^-(2r)$ are

$$2^{2r-2},\ 2^{r-2} \text{ and } -2^{r-2},$$

By Hoffman's bound, an independent set in $S^-(2r)$ has size at most $2^{r-1}-1$ and hence $\chi(S^-(2r))\geqslant 2^r+1$. It is clear that $\mathrm{rk}_2(S^-(2r))=2r$ and therefore $\chi(S^-(2r))=2^r+1$. ∎

THEOREM 5.10. *The graph $N^-(2r)$ has chromatic number 2^r+1.*

Proof. Every maximal totally isotropic subspace of $GF(2)^{2r}$ contains 2^{r-1} singular points and 2^{r-1} non-singular points. Therefore $N^-(2r)$ has independent sets of size at least 2^{r-1}. This follows because the span (in $GF(2)^{2r}$) of an independent set of vertices of $Y(2r)$ is a totally isotropic subspace. Therefore the span of an independent set with at least 2^{r-1} vertices is a maximal totally isotropic subspace, which contains only 2^{r-1} non-singular points. Therefore

$$\chi(N^-(2r))\geqslant\frac{2^{2r-2}+2^{r-1}}{2^{r-1}}=2^r+1$$

and hence $N^-(2r)$ has chromatic number 2^r+1. ∎

The consequence of these two results is that any induced subgraph of $Y(2r)$ that contains either $S^-(2r)$ or $N^-(2r)$ has chromatic number 2^r+1.

6. CONCLUDING REMARKS

The concept of describing a graph as a set of vectors together with an associated form giving adjacency has frequently been used with a variety of different applications. Usually adjacency is represented by the value of the form being zero, so orthogonality graphs are more commonly used than their complements.

The description of a graph as the non-orthogonality graph of a set of vectors with respect to a symplectic form is a "symplectic embedding" in the sense of Garzon [3]. Garzon addressed the question of finding symplectic embeddings of a given "dimension" but did not make it explicit that his dimension is the rank of the adjacency matrix.

The symplectic graphs over $GF(2)$ (and their associated subgraphs) have often been referred to in the literature. In the context of constructing certain Lie algebras, Rotman and Weichsel [12] describe embedding a graph into a symplectic space over $GF(2)$; their approach is essentially equivalent to our development in Section 2. Their aim was to classify certain subsets of symplectic space over $GF(2)$ called "J-systems." A J-system is equivalent to a graph whose complement satisfies a combinatorial regularity property known as the *cotriangle property*; Shult (see [15] and [16]) classified such graphs as being $L(K_n)$, $Y(2r)$ or $N^{\pm}(2r)$. See also Seidel [13] [14] where an almost identical class of subsets is classified, and Kaplansky [5] where a complete list of J-systems is given, but it is unclear as to whether or not the author is claiming a complete classification.

Peeters [8, 9] also uses these graphs, again with a somewhat different application. For example, being interested in the ranks of strongly regular graphs, he observes that the complement of the symplectic graph is characterized by its strong regularity parameters and the minimality of its 2-rank.

The general problem of determining the relationship between the rank and chromatic number of a graph is considerably more complicated if the rank is taken over fields other than $GF(2)$. There is no longer a single universal graph, essentially due to the existence of non-zero elements other than 1. Ellingham [2] observes that there is a finite collection of maximal reduced graphs of any given rank, and that the chromatic number of any graph is bounded above by the largest chromatic number of this finite collection (or more generally, any hereditary property of a graph is so bounded.) He gives an algorithm for constructing the maximal reduced graphs of any given rank, but the number of such graphs grows rapidly with the rank.

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